Uncertainty of Integrated Quantities using Goniometric Data: What to do with the space between the measurements

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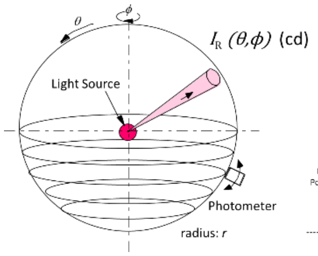
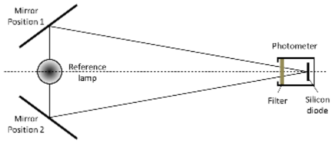
Goniometer measurements





$$\Phi = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I(\theta,\phi)\sin\theta d\theta\, d\phi$$

$$I(\theta,\phi) = ([y(\theta,\phi) - y_d(\theta,\phi)]G(\theta,\phi)H(\theta))/C_I$$

$$C_I = \frac{1}{\Phi_R}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left[y_R(\theta,\phi) - y_{R_d}(\theta,\phi)\right]G(\theta,\phi)H(\theta)\sin\theta d\theta\, d\phi$$

$$y_I = \frac{2\pi}{C_I}\sum_{0}^{\pi}([\bar{y}(\theta)\bar{G}(\theta) - \overline{y_d}(\theta)\bar{G}(\theta)]\cdot H(\theta)\cdot\sin\theta\cdot\Delta\theta)$$

$$\bar{y}(\theta)\bar{G}(\theta) = \frac{1}{n}\sum_{\phi=0}^{2\pi} y(\theta,\phi)G(\theta,\phi)$$

Uncertainty of the solid angle

Confidence band – 95 % of the time the fit falls within a band

$$\pm t\left(n-p, 1-\frac{\alpha}{2}\right)\sqrt{a^TCa} \qquad a = \frac{\partial F}{\partial p}\bigg|x$$

Prediction band – 95 % of the measured points fall within a band

$$\pm t\left(n-p, 1-\frac{\alpha}{2}\right)\sqrt{\chi^2 + a^TCa}$$

Student's t distribution with *n-p* degrees of freedom having probability 1-*α*/2

a vector of partial derivatives of the model with respect to the coefficients evaluated at the given value of the independent variable

C covariance matrix

No analytical solution for a 2-D fit

Solution - Monte Carlo analysis

Correlation Matrix	k_0	k_1	k_2	k_3	k_4	k_5
k_0	1.000	-0.537	-0.570	0.222	0.606	0.301
k_1	-0.537	1.000	0.117	-0.880	-0.402	-0.002
k_2	-0.570	0.117	1.000	0.000	-0.292	-0.928
k_3	0.222	-0.880	0.000	1.000	0.004	0.000
k_4	0.606	-0.402	-0.292	0.004	1.000	0.004
k_5	0.301	-0.002	-0.928	0.000	0.004	1.000

Cholesky	1	2	3	4	5	6
1	1.000	-0.537	-0.570	0.222	0.606	0.301
2	0.000	0.844	-0.225	-0.902	-0.091	0.189
3	0.000	0.000	0.790	-0.096	0.042	-0.904
4	0.000	0.000	0.000	0.357	-0.586	0.049
5	0.000	0.000	0.000	0.000	0.529	-0.178
6	0.000	0.000	0.000	0.000	0.000	0.151

Cholesky decomposition

A well known fact from linear algebra is that any symmetric positive-definite matrix, M, may be written as

$$M = U^TDU$$

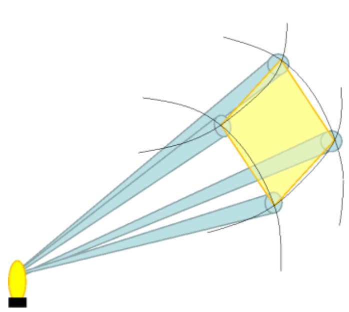
where U is an upper triangular matrix and D is a diagonal matrix with positive diagonal elements. Since our variance-covariance matrix, Σ, is symmetric positive-definite, we can therefore write

$$\Sigma = U^TDU = (U^T\sqrt{D})(\sqrt{D}U) = (\sqrt{D}U)^T(\sqrt{D}U)$$

The matrix $C = \sqrt{D}U$ therefore satisfies $C^TC = \Sigma$. It is called the Cholesky decomposition of Σ.

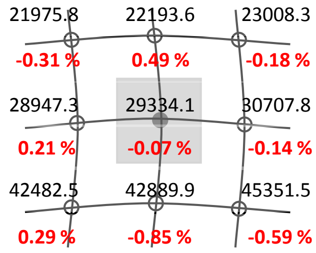
$$\begin{bmatrix} 1.435 \\ 0.582 \\ -0.931 \\ -0.890 \\ -0.074 \\ 0.408 \end{bmatrix} = C^T \times \begin{bmatrix} 1.435 \\ 1.604 \\ 0.313 \\ 0.747 \\ -0.705 \\ -1.366 \end{bmatrix}$$

Weighted solid angle



Mathematical model

$$y(\theta,\phi) = k_0 + k_1\theta + k_2\phi + k_3\theta^2 + k_4\theta\phi + k_5\phi^2$$



- Perform least squares fit to data
- Weighted by relative uncertainty
- Results for 0.10 %

k_0 = 88,457 ± 375
k_1 = -1436.1 ± 405
k_2 = -668,854 ± 4031
k_3 = 3012.6 ± 156
k_4 = -25095 ± 1131
k_5 = 1,741,007 ± 13959

Previous fit chi-square – 161.4, null hypothesis rejected soundly

Null hypothesis – the model represents the distribution described by the measured data points

Chi-square test as a 'goodness of fit' test

Based on a confidence level (α < 0.05)

Degree of freedom (ν = 3, 9 points – 6 parameters)

→ Critical value 7.815

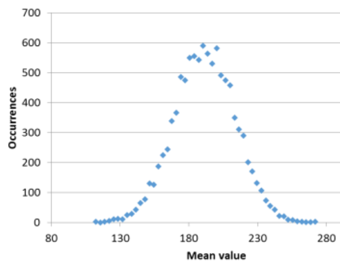
Relative uncertainty = 0.46 %

k_0 = 88,457 ± 1727
k_1 = -1436.1 ± 1863
k_2 = -668,854 ± 18,543
k_3 = 3012.6 ± 720
k_4 = -25095 ± 5205
k_5 = 1,741,007 ± 64,215

Chi-square = 7.63

Center point
191.3 lm
+ Four corner points
196.2 lm
10 x 10 weighted points
192.9 lm
Weighted average – integral
193.4 lm

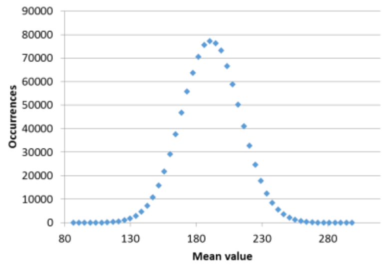
10000 runs



Mean	Sdev
193.52	22.17
193.13	22.02
193.89	22.41
192.98	22.33
193.24	22.30
193.87	21.95
193.57	22.32
193.39	22.36
193.33	22.06
193.54	22.10
193.45	22.20
0.30	0.16
0.15 %	0.73 %

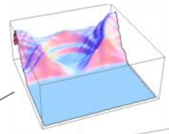
1000000 runs

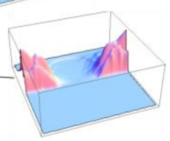
Mean	Sdev
193.38	22.19
193.41	22.19
193.39	22.19
193.38	22.18
193.37	22.21
193.39	22.20
193.41	22.22
193.40	22.19
193.37	22.20
193.36	22.19
193.39	22.20
0.02	0.01
0.01 %	0.05 %

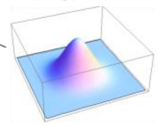


- Four sets of sample data at different angular spacings (θ=1,2,5,10, ϕ=5,22.5)
  - Real data - high intensity
    - Luminous intensity as a function of spherical angle
  - Real data - low intensity
    - Luminous intensity as a function of spherical angle
  - Fake data - constant intensity
    - Luminous intensity at a constant value of 1 candela at every spherical angle
  - Fake data - functional intensity
    - Luminous intensity according to

$$f(\theta,\phi) = \cos\left(\frac{\phi}{2}+\frac{\pi}{2}\right)^5\cos\left(\theta+\frac{\pi}{2}\right)^5$$







Fake data provided accurate results with very little error (<0.01% error for all)

% Uncertainty (68.3% Confidence, 0.1% Relative Uncertainty)				
ϕ \ θ	1	2	5	10
5	1.9	3.3	5.2	6.4
22.5	4.4	7.8	13.8	16.3

Conclusions and concerns

Relationship between model and relative uncertainty between points
- apply bicubic spline, uncertainty in matching derivatives

Correlation between solid angle determinations

Effect on the industry